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Interactive Probability Models: Inverse Problems on the Sphere

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We expose a class of probabilistic models with only two outcomes that we call interactive probability models for the analysis of data that arise in situations where there is influence of the measurer on the measured. We reconstruct a Borel measure corresponding to possible sets of probabilities that are related to outcomes of experiments. We give three examples: one that corresponds to the quantum mechanical case, one to a deterministic measurement, and one to a situation where the outcome of the measurement is determined by the measurement apparatus only.

1. INTRODUCTION

It is well known that in Kolmogorovian probability theory the origin of the probabilities is due to a lack of knowledge about the precise state of the entity under observation. This scheme turns out to be too narrow to contain quantum mechanics. Indeed, it was shown that the introduction of so-called hidden variables into quantum mechanics allows for dispersionless states and that, in restoring the Kolmogorovian character, we simultaneously destroy the quantum character. Another approach was formulated (D. Aerts, 1986, 1987) where the quantum character is due to a lack of knowledge about the precise interaction between the entity that is being measured and the measurement apparatus. A model for a measurement with only two outcomes was introduced that could generate a very broad spectrum of probabilities. It was shown, for example, that one can generate the same probabilities as a two dimensional Hilbert-space model, and, by varying a parameter called (D. Aerts et al., 1993), one is able to recover probabilities that one can identify as being derivable within a standard Kolmogorovian framework (S. Aerts, 1996). The parameter $\in$ in this model was introduced ad hoc: it

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produced the correct probabilities and could be interpreted as a measure of the interaction between the apparatus and the entity under observation. In this article we want to follow an opposite strategy: given the probabilities and the sphere model, what can we deduce about the underlying interactional measure? In order to do so, we first present a slightly modified version of the sphere model.

2. THE MODEL

Let the entity we wish to study be represented by a point on the unit sphere. The experimenter wishes to obtain information about the location of the entity on the sphere. Because we want to model different types of lack of knowledge, we will severely restrict the informational access the experimenter has to the experimental setup. The only way he can obtain knowledge about the entity is to resort to the following experiment. He attaches an elastic on two diametrically opposite points (which he might call up and down) of the sphere in the direction u. The entity (which is in a direction we will denote by the vector v) is then projected orthogonaly onto the elastic. Then the elastic breaks somewhere, dragging the entity along the direction u to one of the two endpoints ($+u$ or $-u$) of the elastic. If the entity is dragged towards the upper point $+u$, the experimenter gives the experiment e_u the outcome $+1$, otherwise he gives the experiment the outcome -1. The probability $P(e_u = +1|v)$ that the above experiment leads to the result $+1$ for a measurement where the elastic is in the direction u and the entity in the direction u equals the probability $P(\tau)$ that the elastic breaks between -1 and the projection $u \cdot v = \tau \in [-1,+1]$ and as such is a measure

$$\tau \mapsto P(\tau) = \mu([-1, \tau]) \tag{1}$$

3. RECONSTRUCTION OF AN UNDERLYING BOREL MEASURE

Given the probability $P(e_u = +1|v) = P(\tau)$ of the outcome $+1$ of an experiment e_u if the state of the entity is characterized by the vector v, the question we want to raise is to what extent this probability $P(\tau)$ determines the Borel measure μ on the interval $[-1,+1]$ (which represents the elastic). To answer this question we will use the Lebesgue decomposition of $P(\tau)$ (see, for example, Rao, 1987), which states that any bounded monotone function P: $\mathbb{R} \to \mathbb{R}$ can be uniquely expressed as

$$P(\tau) = P_a(\tau) + P_s(\tau) + P_d(\tau), \qquad \tau \in \mathbb{R} \tag{2}$$

where P_a is an absolutely continuous, monotone increasing function, P_s a singularly continuous function, and P_d is discontinuous increasing, i.e.,

$P_d(\tau) = \Sigma_{i<\tau}\, a_i$. Moreover, P_a has at most a countable number of nonzero terms and is absolutely convergent. Note that this decomposition of the probability corresponds to a decomposition of the underlying Borel set such that P_a corresponds to point measures, P_s accounts for a thin subset (i.e., has measure zero) of the Borel set, and P_a corresponds to a density function in the following sense: according to the Lebesgue–Vitali theorem, a function f: $[a, b] \to \mathbb{R}$ is absolutely continuous iff it admits an integral representation

$$f(x) = f(a) + \int_a^x f'(t)\, dt \tag{3}$$

where f' is the derivative of f which exists at almost all points of $[a, b]$ and is Lebesgue integrable on the interval $[a, b]$. In our case P_a is absolutely continuous and we will call $P_a'(\tau) = \rho(\tau)$ the density function corresponding to $P(\tau)$. Since P_d has at most a countable number of nonzero terms and the derivative of a singular function is zero almost everywhere by definition, we have

$$P'(\tau) = P_a'(\tau) = \rho(\tau) \tag{4}$$

for almost all $\tau \in [-1,+1]$.

3.1. Some Examples

We will apply the above decomposition to three types of measurement. The probabilities that one encounters in these three situations are all bounded and monotone and thus we can apply the Lebesgue decomposition theorem (2).

1. The first type of measurement we want to model is what we could call an observation. For a measurement to be an observation, we require that it be deterministic, that is, the probability related to an outcome equals one or zero. We will cut the sphere into two hemispheres S_1 and S_2 by means of a plane that is perpendicular to the line through $+u$ and $-u$ and that intersects this line at a point that can fixed with a parameter $\gamma \in [-1,+1]$. Let S_1 be the hemisphere that contains $+u$. We will assume that $P(e_u = +1|v)$ shifts from zero to one as soon as the projection of the entity on the elastic exceeds γ, that is, if $\tau > \gamma$. In this case the outcome $e_u = +1$ really means the entity was in the hemisphere S_1, explaining the name 'observation.' With $h(x)$ the Heaviside or unit step function $[h(x) \equiv 0$ iff $x < 0$, $h(x) \equiv 1$ iff $x \geq 0]$, we can write

$$P(\tau) = P_d(\tau) = h(\tau - \gamma) \tag{5}$$

According to equation (4), $P'(\tau) = 0$ almost everywhere. Sine $P_s = P_a = 0$ in this case, the only candidate for μ is a point measure located at γ:

$$\mu = \delta(\tau - \gamma) \tag{6}$$

2. For the second example we turn to the following question: what possible Borel measure allows us to recover the quantum mechanical probability related to a spin measurement of a spin-1/2 entity prepared in a state v, measured with a Stern–Gerlach apparatus that makes an angle α with v, as

$$P(e_u = +1|v) = \cos^2\left(\frac{\alpha}{2}\right) = \frac{1 + \cos(\alpha)}{2} \tag{7}$$

With $\tau = \cos(\alpha)$, we can relate the probability in equation (7) to τ:

$$P(\tau) = P_a(\tau) = \frac{1 + \tau}{2} \tag{8}$$

with $[a,b] = [-1,+1]$ and $f(a) = 0$, we can apply the Lebesgue–Vitali theorem (3). We see that the probability is an absolute continuous function of τ, hence it allows for a density function:

$$\mu = P'(\tau) = \frac{1}{2} \tag{9}$$

The fact that a constant density function equal to 1/2 reproduces the quantum mechanical probability has long been known (D. Aerts, 1986). We have now proven that it is the *only* density (up to a Borel set with Lebesgue measure zero) on $[-1,+1]$ that generates the probabilities of a two-dimensional Hilbert space model.

3. The third type of measurement could be called a solipsistic measurement because we demand that the results be determined by the apparatus or, equivalently, that the probability is independent of the state of the entity for almost all τ. Thus, with $a \in [0,1]$ we have $P(e_u = +1|v) = a$ for $v \neq |u|$, $P(e_u = +1|v) = 1$ for $v = u$, and $P(e_u = +1|v) = 0$ for $v = -u$. According to equation (4), the solution is zero almost everywhere, except for the endpoints of the elastic. Hence, $P(\tau) = P_a(\tau)$ and the only allowable μ is concentrated in two point measures: one located at -1 and one at $+1$, or, with the use of the Dirac distribution,

$$\mu = a \cdot \delta(\tau + 1) + (1 - a) \cdot \delta(\tau - 1) \tag{10}$$

4. CONCLUSION

We have reconstructed a Borel measure corresponding to the outcome probabilities as a function of the projection. We have given three types of measurement this model can handle: an observation where the result of the measurement is only dependent on the state of the entity; a solipsistic measurement, where the result is only dependent on the measurement apparatus, and the quantum measurement, where the outcome is dependent on both the state and the apparatus.

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